

An Evolving model of online bipartite networks

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Abstract. - Understanding the structure and evolution of online bipartite networks is a significant task since they play a crucial role in various e-commerce services nowadays. Recently, various attempts have been tried to propose different models, resulting in either power-law or exponential degree distributions. However, many empirical results show that the user degree distribution actually follow a stretched exponential decay, which cannot be fully describe by previous models. In this Letter, we propose an evolving model, considering two different user behaviors: random and preferential attachment. Extensive empirical results on two real bipartite networks, *Delicious* and *CiteULike*, show that the theoretical model can well characterize the structural of real networks for both user and object degree distributions. In addition, we introduce a structural parameter, p , to demonstrate that the hybrid user behavior leads to the stretched exponential degree distribution, and the region of power-law tail will increase with the increment of p . The proposed model might shed some lights in understanding the underlying laws governing the structure of real online bipartite networks.

Introduction. – The past decade has witnessed a great explosion of studying and understanding the underlying mechanisms of various real-life networks, ranging from the Internet, scientific collaboration networks, protein networks to social networks, etc [1–7]. Although they respectively have their own properties and characteristics, empirical analyses show that many common characteristics and phenomena can be discovered from networks with such a wide-range functions, e.g. a small average distance between nodes, a large clustering coefficient [8], power-law degree distribution [9] and community structures [10] of the emerging structure. Recently, studies on the mathematics of networks have been driven largely by those observed empirical properties of real networks, as well as network dynamics. However, many pioneering works in this area focus on designing evolutionary models of unipartite networks which only have one kind of nodes, such as Erdős-Rényi network [11], Watt-Strogatz network [8], Barabási-Albert network [9], as well as many extensive variants considering different factors (e.g. aging effect [12, 13] and so-

cial impact [14–16]). Recently, with the advent of Web 2.0 and affiliated applications, the family of *Networks* also has received many new members. One example is the bipartite network which involves two different kinds of nodes with different functions [17–19]. Different from traditional networks, the nodes in a pure bipartite network can be divided into two independent communities, where edges are only allowed to exist between different communities. Nowadays, this bipartite network is widely applied in both online platforms (e.g. online services where users view/purchase products [20–22], or listen to music [23]), biology [24–27] and medical science [28–30] and theoretical studies [31–34]. There is also a vast class of researches that have recently reported many universal properties in unipartite networks, such as power-law degree distribution and correlation [17, 19] and community structure [34–38], could also be found in bipartite networks. Consequently, it has attracted an increasing attention from scientific community due to its wide application and bright prospect in characterizing the essential properties of real networks. The first and natural attempt is to project the bipartite network to a corresponding unipartite network and using

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methods for traditional networks [39–42]. However, it is argued that such one-mode projection ignores much informative structure and relationship, subsequently, it would give unreliable or incorrect results [36, 43, 44]. Therefore, a more common approach is to keep the original bipartite structure, investigate both its specific and common properties, and try to uncover the underlying mechanism driving the emergence of this two-mode network. Newman *et al.* used the random graph model to describe social networks of both unipartite and bipartite relations [43]. Using generating functions [45], they concluded that the clustering and average degree of real affiliation networks, as one typical kind of bipartite networks, agreed well with the theoretical prediction. Lambiotte *et al.* proposed a personal identification and community imitation (PICI) based model to consider both effects of collective behavior and personalization [23]. This model generated an exponential and power-law degree distribution for music groups and owners, respectively. Sood and Redner introduced the voter model on networks of power-law degree distributions with and without degree correlation, both of which showed the consensus time was greatly dependent on the value of exponent [46]. Noh *et al.* demonstrated that different mechanisms would generate different shape of degree distributions in group selection systems [47]. That is to say, a random selection process would result in an exponential distribution of the activity degree, otherwise a power-law distribution of group size and activity degree would arise from the resultant force of preferential selection and fixed-probability creation. Sneppen *et al.* proposed a minimalistic model of directed bipartite network, and a self-organization phenomenon was observed by a dynamical reconnection process [48]. Similar result was also found in collaboration bipartite networks via preferential attachment of actors' degree [44]. Hence, this model only reproduced that one kind of node followed power-law but neglecting outputs of the other side of nodes. Saavedra *et al.* introduced two mechanisms, specialization and interaction, would produce exponential degree distribution for both sides [49]. In addition, they found this bipartite cooperation can well characterize the structure of both ecological and organization networks.

In this Letter, we focus on studying the degree distribution of online bipartite networks where users view/choose/select objects (e.g. bookmarks, music, movies), as well as the underlying mechanisms. Despite many previous studies demonstrated that both exponential and power-law degree distribution could be obtained by corresponding models, empirical analysis of online bipartite networks shows that the user degree distribution follows stretched exponential instead of pure exponential decay, while the object degree distribution always obeys power-law [19, 50], and it can not be fully explained by previous models. Therefore, We propose an evolutionary model to consider the proactive selection activity of users and the passive pattern of objects. Theoretical analysis shows that the present model can not only well reproduce

the two different degree distribution, but also find good agreements of two real-world data sets, *Delicious*¹ and *CiteULike*². In addition, we find that the structural parameter, p , determines the transformation from exponential to power-law decay of the user degree distribution.

Model. – In this section, we shall propose an evolving model to uncover the growing dynamics of online bipartite networks. Here, we mainly consider two mechanisms: random and preferential attachment. In particular, we assume there are two kind of online behaviors for users: she can either randomly choose an object or pick up an item according to its popularity. On one hand, considering a new user involving in the system, it would be difficult for her to select a suitable object from numerous candidates. One reasonable action she would take is to choose a popular item since other users also like it. On the other hand, old users who have devoted much time in playing the online platform, would know to find their own favorites and thus are likely to select personalized (hence might be less popular) items. That is to say, users are very proactive in performing online activities. In [51–53], they reported such a hybrid behavior would result in a mixture between power-law and exponential distribution. By contrast, objects in online systems are always in a passive pattern, hence do not have any choice but waiting to be selected to gain popularity. Therefore, we assume objects always grow based on preferential attachment in our model.

We begin our study with some related definitions of bipartite graph that we will analyze. The bipartite graph can be represented by $G = (U, O, E)$, where U and O are two disjoint sets of nodes, respectively representing users and objects, and $E \subseteq U \times O$ is the set of edges. The difference with classical graph lies in the fact that edges exist only between user vertices and object vertices. The model starts from an initial bipartite network: there exist u_0 nodes in U , o_0 nodes in O and e_0 edges in set E . Given a user i in U and an object j in O , denote k_i as the degree of i and l_j as the degree of j in the bipartite network. Then, $e_0 = \sum_i k_i = \sum_j l_j$ ($k_i, l_j \geq 1, i = 1, 2, \dots, u_0, j = 1, 2, \dots, o_0$). There are totally $N = u_0 + t$ users and $M = o_0 + t$ objects in the model at time t . Consequently, the model can be described as following:

- adding a new user: Connect the new user node to m different nodes already in O by preferential probability $\frac{l_j}{\sum_{j=1}^{o_0+t-1} l_j}$.
- adding a new object: Link the new node to n different nodes already in U by preferential probability $\frac{k_i}{\sum_{i=1}^{u_0+t-1} k_i}$.

¹<http://www.delicious.com/>

²<http://www.citeulike.com>

- edges evolving randomly: Two kinds of old nodes are connected by c edges, which are chosen as: users are selected randomly with probability $\frac{1}{u_0+t}$, while objects in O are selected by preferential probability $\frac{l_j}{\sum_{j=1}^{o_0+t-1} l_j}$.
- edges evolving by preferential attachment: Two kinds of old nodes are connected by b edges, which are chosen as: users are selected by preferential probability $\frac{k_i}{\sum_{i=1}^{u_0+t-1} k_i}$, and objects are also selected by preferential probability $\frac{l_j}{\sum_{j=1}^{o_0+t-1} l_j}$.

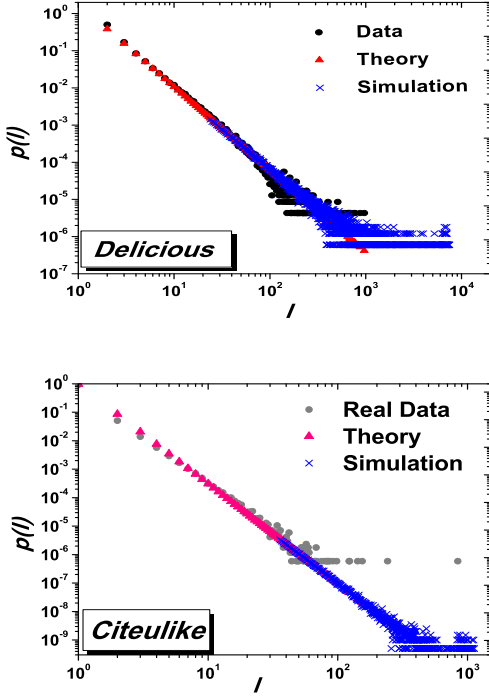


Fig. 1: (Color online) Object degree distribution in a log-log scale of *Delicious* and *Citeulike* for (a) Real Data, (b) Theory, and (c) Simulation.

Analytical Analysis. –

Object degree distribution. From the aforementioned model description, we can write the dynamics of degree for object O_j

$$\frac{\partial l_j}{\partial t} = m \frac{l_j}{\sum_{j=1}^{o_0+t-1} l_j} + c \frac{l_j}{\sum_{j=1}^{o_0+t-1} l_j} + b \frac{l_j}{\sum_{j=1}^{o_0+t-1} l_j}, \quad (1)$$

where $\sum_{j=1}^{o_0+t-1} l_j = \langle l \rangle M$, $M = o_0 + t$, $\langle l \rangle = \frac{(m+n+c+b)t+e_0}{o_0+t}$. Then Eq. 1 is approximated to

$$\frac{\partial l_j}{\partial t} = \frac{wl_j}{vt}, \quad (2)$$

where $w=m+c+b$, $v=m+n+c+b$, $t \gg m, n, c, b$ and $i = 1, 2, \dots, t$.

The initial degree of node j satisfies $l_j(t_j) = n$, where t_j represents the time that node j is added into O . Therefore we obtain following equation by solving Eq. 2 [1]

$$l_j(t) = n \left(\frac{t}{t_i} \right)^{\frac{w}{v}} \quad (3)$$

Let $l_j(t) < l$, then $t_i > t \left(\frac{n}{l} \right)^{\frac{v}{w}}$. So the cumulative probability $P(l_j(t) < l)$ can be denoted by $P(t_i > t \left(\frac{n}{l} \right)^{\frac{v}{w}})$, such that

$$P(l_j(t) < l) = P(t_i > t \left(\frac{n}{l} \right)^{\frac{v}{w}}). \quad (4)$$

In the model, all nodes are added into network with the same time interval, which means

$$p(t_j) = \frac{1}{o_0 + t}. \quad (5)$$

Integrating Eq. 4 and Eq. 5, we can obtain the cumulative probability

$$p(l_j(t) < l) = p(t_j > t \left(\frac{n}{l} \right)^{\frac{v}{w}}) = 1 - \frac{t}{o_0 + t} \left(\frac{l}{n} \right)^{-\frac{v}{w}}. \quad (6)$$

Finally, with assuming as $t \gg m, n, c, b$, the object degree distribution can be written

$$p(l) = \frac{\partial p(l_j(t) < l)}{\partial l} \approx \frac{v}{w} n^{\frac{v}{w}} l^{-\frac{v}{w}-1}. \quad (7)$$

From Eq. 7, it is can be found that the object degree distribution accords with power-law distribution, with exponent $\gamma_l = 1 + \frac{v}{w}$.

User degree distribution. Similar to the theoretical analysis of object degree distribution, the dynamics of user u_i can be written as

$$\frac{\partial k_i}{\partial t} = n \frac{k_i}{\sum_{i=1}^{u_0+t-1} k_i} + c \frac{1}{N} + b \frac{k_i}{\sum_{k=1}^{u_0+t-1} k_i}, \quad (8)$$

where $\sum_{i=1}^{u_0+t-1} k_i = \langle k \rangle N$, $N = u_0 + t$, $\langle k \rangle = \frac{(m+n+c+b)t+e_0}{u_0+t}$. Then Eq. 8 is approximated to

$$\frac{\partial k_i}{\partial t} = \frac{uk_i}{vt} + \frac{c}{N}, \quad (9)$$

where $u=n+b$, $v=m+n+c+b$, $t \gg m, n, c, b$ and $i = 1, 2, \dots, t$.

Since the initial degree of all users satisfies $k_i(t_i) = m$, where t_i represents the time user u_i is added into U . Then we get following equation by solving Eq. 9

$$k_i(t) = \frac{\left(\frac{t}{t_i} \right)^{\frac{u}{v}} (cv + mu) - cv}{u}. \quad (10)$$

Substitute $p(t_i) = \frac{1}{u_0+t}$ into Eq. 10, we will get the cumulative probability

$$p(k_i(t) < k) = 1 - \frac{t}{u_0+t} \left(\frac{cv+ku}{cv+mu} \right)^{-\frac{v}{u}}. \quad (11)$$

So the user degree distribution function is finally achieved by assuming $t \gg m, n, c, b$

$$p(k) = \frac{\partial p(k_i(t) < k)}{\partial k} \approx (cv+mu)^{\frac{v}{u}} v (cv+ku)^{-\frac{v}{u}-1}. \quad (12)$$

From Eq. 12, we know that the user degree distribution is a mixture of exponential and power-low forms [51–53], which is now familiar as stretched exponential distribution [54].

Results & Analysis. – In this section, we use two data sets to evaluate the proposed model. The first one is *Delicious*, one of the most popular social bookmarking web sites, which allows users not only to store and organize personal bookmarks, but also to look into users' collection and find what they might be interested in [55]. The other is from *Citeulike*, which also has similar characterizations with *Delicious*. Table. 1 shows the basic statistical properties of the two data sets.

Degree distributions. Fig. 1 reports the object degree distribution result. It can be seen that both the simulation and analytical results fit well with the real data. In addition all the object-degree distributions are power-law, as $p(l) = l^{-\gamma}$, with $\gamma = 3.50$ and 2.22 for *Delicious* and *CiteULike*, respectively.

For the user degree distribution, we focus on the cumulative degree distribution. Fig. 2 illustrates the cumulative degree distribution for users. Again, we find good agreements among the simulation, analytical and empirical results, in particular at the tail of the distribution. Therefore, the present model can qualitatively accurately model the general real-world networks by assuming users' mixture behavior. The degree distributions for all users are similar to stretched exponential distribution $p(k) \sim e^{-(\frac{k}{k_0})^{c_0}}$, $0 < c_0 < 1$.

Table 1: Basic statistical properties of the *Delicious* and *Citeulike*. $|U|$, $|O|$ and $|E|$ denote the number of users, objects and edges, respectively. $\rho = \frac{|E|}{|U||O|}$ denotes the sparsity of the data.

Data set	$ U $	$ O $	$ E $	ρ
<i>Delicious</i>	9,998	232,657	123,995	5.305×10^{-4}
<i>Citeulike</i>	42,801	397,536	7,083,253	4.163×10^{-4}

Understanding the effects of random and preferential attachment. From analysis in estimation of network, the user degree distribution is determined together by both preferential and random linking mechanisms. In order to further understand the effects of these two mechanisms,

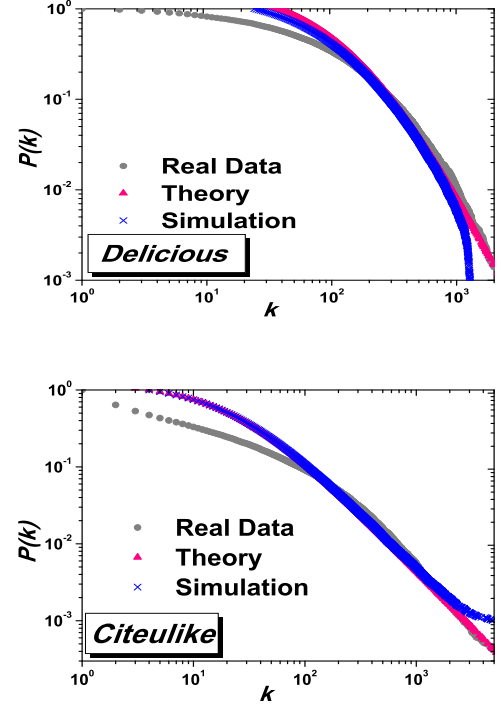


Fig. 2: (Color online) The cumulative degree distribution of users in a log-log scale of *Delicious* and *Citeulike* for (a) Real Data, (b) Theory, and (c) Simulation.

we introduce a structural parameter, $p \in [0, 1]$, to quantify different weights of them. Denote p as the weight of preferential mechanism, and $1 - p$ refers to random choosing mechanism. According to the model description, we have $p = \frac{n+b}{n+b+c}$. Fig. 3 shows theoretically the user cumulative degree distribution for different p .

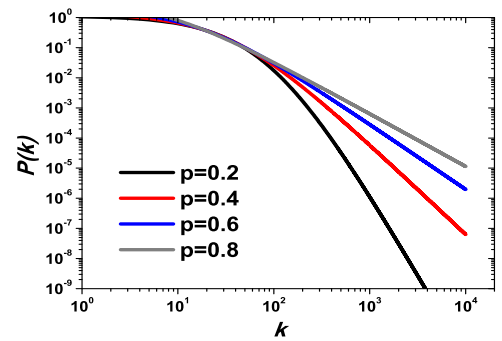


Fig. 3: (Color online) The theoretical cumulative user degree distribution in log-log scale for different p , including (a) $p=0.2$ (black); (b) $p=0.4$ (red); (c) $p=0.6$ (blue); (d) $p=0.8$ (green).

As shown in Fig. 3, an obvious correlation between p and the user cumulative degree distribution is observed. In addition, the scale-free region increases with the increment

of p , which indicates that p indeed can characterize the different structures driven by the two mechanisms. That is to say, for the extreme cases, $p = 1$ will produce a pure scale-free degree distribution, and $p = 0$ will generate an exponential degree distribution. Otherwise, a stretched exponential decay will be observed for $p \in (0, 1)$. The cumulative form of stretched exponential distribution is $P(k) \sim e^{-(\frac{k}{k_0})^{c_0}}$, where k_0 is a constant and $0 \leq c_0 \leq 1$ is the characteristic exponent. The scale-free region of $P(k)$ increases with the decrement of c_0 . Obviously, there may be a positive correlation between $1 - p$ and c_0 , such as $1 - p \sim c$. The exponent c can be determined by considering the cumulative distribution $P(k) \sim e^{-(\frac{k}{k_0})^{c_0}}$, which can be rewritten as $\log(-\log P(k)) \sim c_0 \log k$ [19]. After if the corresponding curve can be well fitted by a straight line, then the slope is $c = \alpha(1 - p)$, where α is a scale factor, Fig. 4 reports this result.

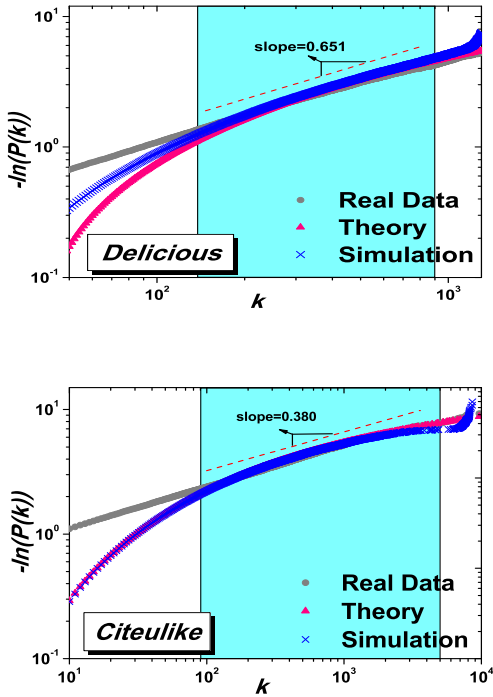


Fig. 4: (Color online) The cumulative degree $P(k)$ versus k with log-log scale for *Delicious* and *Citeulike*. The corresponding curve can be well fitted by a straight line and the slope is c_0 .

Conclusions and Discussion. Previous models about evolving bipartite networks usually lead to power-low degree distribution for both of users and objects, which conflicts with the properties of some real networks, of which user degree distribution is stretched exponential distribution. In this Letter, we propose an evolving model, trying to characterize the user behaviors. The proposed model considers that users' actions are determined by both random and preferential mechanisms, and objects are selected

mainly by preferential mechanism. Results of real data, theory and simulation are well fitted with each other. In addition, we also compare the weights of the two different mechanisms, and find out that a clear correlation between the structural parameter and the shape of user cumulative degree distribution. Our proposed model might shed some lights in understanding the underlying laws governing the structure of real online bipartite networks.

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REFERENCES

- [1] ALBERT R. and BARABÁSI A.-L., *Rev. Mod. Phys.*, **74** (2002) 47.
- [2] DOROGVTSEV S. N. and MENDES J. F. F., *Adv. Phys.*, **51** (2002) 1079.
- [3] NEWMAN M. E. J., *SIAM Rev.*, **45** (2003) 167.
- [4] BOCCALETTI S., LATORA V., MORENO Y., CHAVEZ M. and HUANG D.-U., *Phys. Rep.*, **424** (2006) 175.
- [5] COSTA L. D. F., RODRIGUES F. A., TRAVIESO G. and BOAS P. R. U., *Adv. Phys.*, **56** (2007) 167.
- [6] ARENAS A., DÍAZ-GUILERA A., KURTHS J., MORENO Y. and ZHOU C., *Phys. Rep.*, **469** (2008) 93.
- [7] CASTELLANO C., FORTUNATO S. and LORETO V., *Rev. Mod. Phys.*, **81** (2009) 591.
- [8] WATTS D. J. and STROGATZ S., *Nature*, **393** (1998) 440.
- [9] BARABÁSI A.-L. and ALBERT R., *Science*, **286** (1999) 509.
- [10] GIRVAN M. and NEWMAN M. E. J., *Proc. Natl. Acad. Sci. U.S.A.*, **99** (2002) 7821.
- [11] ERDŐS P. and RÉNYI A., *On the evolution of random graphs* (Akad. Kiadó) 1960.
- [12] DOROGVTSEV S. N. and MENDES J., *Phys. Rev. E*, **62** (2000) 1842.
- [13] DOROGVTSEV S. and MENDES J., *EPL*, **52** (2007) 33.
- [14] JIN E., GIRVAN M. and NEWMAN M., *Phys. Rev. E*, **64** (2001) 016101.
- [15] NEWMAN M. and PARK J., *Phys. Rev. E*, **68** (2003) 036122.
- [16] CASTELLANO C., FORTUNATO S. and LORETO V., *Rev. Mod. Phys.*, **81** (2009) 591.
- [17] PELTOMÄKI M. and ALAVA M., *J. Stat. Mech.*, **2006** (2006) P01010.
- [18] GOLDSTEIN M., MORRIS S. and YEN G., *Phys. Rev. E*, **71** (2005) 026108.
- [19] SHANG M.-S., LÜ L., ZHANG Y.-C. and ZHOU T., *EPL*, **90** (2010) 48006.
- [20] WANG D., ZHOU L. and DI Z. D., *Physica A*, **363** (2006) 359.
- [21] PERUANI F., CHOUDHURY M., MUKHERJEE A. and GANGULY N., *EPL*, **79** (2007) 28001.
- [22] HIDALGO C. and HAUSMANN R., *Proc. Natl. Acad. Sci. U. S. A.*, **106** (2009) 10570.

- [23] LAMBIOTTE R. and AUSLOOS M., *Phys. Rev. E*, **72** (2005) 066107.
- [24] ERGÜN G., *Physica A*, **308** (2002) 483.
- [25] AITTOKALLIO T. and SCHWIKOWSKI B., *Brief. Bioinform.*, **7** (2006) 243.
- [26] MA'AYAN A., JENKINS S., GOLDFARB J. and IYENGAR R., *Mt. Sinai J. Med.*, **74** (2007) 27.
- [27] GOH K., CUSICK M., VALLE D., CHILDS B., VIDAL M. and BARABÁSI A.-L., *Proc. Natl. Acad. Sci. U.S.A.*, **104** (2007) 8685.
- [28] YILDIRIM M., GOH K., CUSICK M., BARABÁSI A.-L. and VIDAL M., *Nat. Biotechnol.*, **25** (2007) 1119.
- [29] NACHER J. and SCHWARTZ J., *BMC Pharmacol.*, **8** (2008) 5.
- [30] BARABÁSI A., GULBAHCE N. and LOSCALZO J., *Nat. Rev. Genet.*, **12** (2011) 56.
- [31] STROGATZ S., *Nature*, **410** () 268.
- [32] WATTS D., *Ann. Rev. Sociol.*, (2004) 243.
- [33] GUILLAUME J. and LATAPY M., *Infor. Process. Lett.*, **90** (2004) 215.
- [34] MUCHA P., RICHARDSON T., MACON K., PORTER M. and ONNELA J., *Science*, **328** (2010) 876.
- [35] LIND P., GONZALEZ M. and HERRMANN H., *Phys. Review E*, **72** (2005) 056127.
- [36] GUIMERÀ R., SALES-PARDO M. and AMARAL L., *Phys. Rev. E*, **76** (2007) 036102.
- [37] BARBER M., *Phys. Rev. E*, **76** (2007) 066102.
- [38] ZHANG P., WANG J., LI X., LI M., DI Z. and FAN Y., *Physica A*, **387** (2008) 6869.
- [39] LAMBIOTTE R. and AUSLOOS M., *Phys. Rev. E*, **72** (2005) 066117.
- [40] KOSSINETIS G., *Social Netw.*, **28** (2006) 247.
- [41] LEHMANN S., SCHWARTZ M. and HANSEN L., *Phys. Rev. E*, **78** (2008) 016108.
- [42] MUKHERJEE A., CHOUDHURY M. and GANGULY N., *Physica A*, **390** (2011) 3602.
- [43] NEWMAN M., WATTS D. and STROGATZ S., *Proc. Natl. Acad. Sci. U. S. A.*, **99** (2002) 2566.
- [44] RAMASCO J., DOROGOVTSSEV S. and PASTOR-SATORRAS R., *Phys. Rev. E*, **70** (2004) 036106.
- [45] NEWMAN M. E. J., STROGATZ S. H. and WATTS D. J., *Phys. Rev. E*, **64** (2001) 026118.
- [46] SOOD V. and REDNER S., *Phys. Rev. Lett.*, **94** (2005) 178701.
- [47] NOH J., JEONG H., AHN Y. and JEONG H., *Phys. Rev. E*, **71** (2005) 036131.
- [48] SNEPPEN K., ROSVALL M., TRUSINA A. and MINNHAGEN P., *EPL*, **67** (2007) 349.
- [49] SAAVEDRA S., REED-TSOCHAS F. and UZZI B., *Nature*, **457** () 463.
- [50] ZHANG Z.-K. and LIU C., *J. Stat. Mech.*, (2010) P10005.
- [51] LIU Z., LAI Y., YE N. and DASGUPTA P., *Phys. Lett. A*, **303** (2002) 337.
- [52] ZHANG Z., RONG L., WANG B., ZHOU S. and GUAN J., *Physica A*, **380** (2007) 639.
- [53] TIAN L., HE Y., LIU H. and DU R., *Phys. Lett. A*, **376** (2012) 1827.
- [54] LAHERRÉRE J. and SORNETTE D., *Eur. Phys. J. B*, **2** (528) 1998.
- [55] ZHANG Z.-K., ZHOU T. and ZHANG Y.-C., *Physica A*, **389** (2010) 179.